

Operator extension of strong subadditivity of entropy

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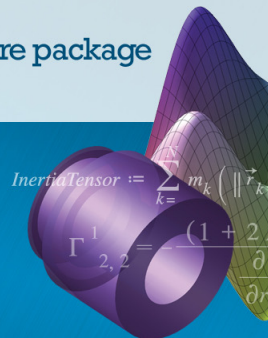
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$$InertiaTensor := \sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{I} - \vec{r}_k \vec{r}_k^T \right)$$

$$\Gamma_{2,2}^1 = -\frac{(1+2)}{\partial r}$$

Operator extension of strong subadditivity of entropy

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We prove an operator inequality that extends strong subadditivity of entropy: after taking a trace, the operator inequality becomes the strong subadditivity of entropy.

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I. INTRODUCTION

There are inequalities satisfied by Von Neumann entropy, such as non-negativity and strong subadditivity.^{1,2} Strong subadditivity plays an important role in quantum information theory: it is essential in virtually all nontrivial coding theorems. Strong subadditivity has also found applications to geometric entropy,³ conformal field theory (CFT),⁴ and topological entanglement entropy.⁵

In this paper, we prove a certain operator inequality that extends strong subadditivity. Motivation for studying such inequality comes from recent observation of Li and Haldane (LH).⁶ LH conjectured and numerically substantiated that eigenvalues of reduced density matrix contains universal information about the quantum phase. While their result concerns a variational wavefunction for a fractional quantum Hall system, author was able to reproduce some of the consequences without resorting to the underlying structure of the wavefunction dictated by the CFT.⁷

One important assumption was that certain operator generalization of strong subadditivity holds for general quantum states. Specifically, the author conjectured the following inequality,

$$\text{Tr}_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0, \quad (1)$$

where \hat{H}_A is formally defined as

$$\hat{H}_A = -I_{A^c} \otimes \log(\rho_A). \quad (2)$$

Other operators are defined similarly. Whenever logarithm of reduced density matrix appears, tensor product with identity for the rest of the subsystems is implicitly assumed. We prove the conjecture in this paper.

II. OPERATOR GENERALIZATION OF STRONG SUBADDITIVITY

Since Lieb and Ruskai's seminal result,¹ alternative proofs for strong subadditivity have been introduced by several authors.^{8–11} In particular, Effros recently presented a proof based on perspective of operator convex function.¹¹ Effros basically extended the notion of perspective function from real numbers to operators. Given a function f , perspective of f is defined as

$$g(x, t) = f(x/t)t. \quad (3)$$

If $f(x)$ is convex, $g(x, t)$ is jointly convex in x and t . Main insight of Effros is that similar statement holds for function f that is *operator convex*. To be more precise, he showed the following statement.

Theorem 1: (Effros¹¹) *If $f(x)$ is operator convex, and $[L, R] = 0$, perspective*

$$g(L, R) = f(L/R)R \quad (4)$$

is jointly convex in the sense that if $L = cL_1 + (1 - c)L_2$ and $R = cR_1 + (1 - c)R_2$ with $[L_i, R_i] = 0$ ($i = 1, 2$), $0 \leq c \leq 1$,

$$g(L, R) \leq cg(L_1, R_1) + (1 - c)g(L_2, R_2). \quad (5)$$

We choose a matrix algebra $\mathcal{B}(\mathcal{H})$ with an inner product structure of $\langle X, Y \rangle = \text{Tr}(XY^\dagger)$, where X, Y are $n \times n$ matrices. Following Effros, we choose L and R to be superoperators that multiplies matrix from left or right. For $X \in \mathcal{B}(\mathcal{H})$, L and R are defined as follows.

$$\begin{aligned} LX &= \rho X, \\ RX &= X\sigma. \end{aligned} \quad (6)$$

L and R commute with each other. One can also show the following relations.

$$\begin{aligned} \log(L)X &= \log(\rho)X, \\ \log(R)X &= X\log(\sigma). \end{aligned} \quad (7)$$

Effros' result implies the following statement.

Theorem 2:

$$\text{Tr}_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0. \quad (8)$$

Proof: Let $f(x) = x \log x$. Since $f(x)$ is operator convex,¹² $g(L, R) = L \log(L) - L \log(R)$ is jointly convex in L and R . Therefore,

$$\langle g(L, R)(O), O \rangle = \text{Tr}(\rho \log \rho O O^\dagger - \rho O \log \sigma O^\dagger) \quad (9)$$

is jointly convex in L and R for all $O \in \mathcal{B}(\mathcal{H})$. Joint convexity of Eq. (9) was originally proved by Petz.¹⁴ Choose $\rho = \rho_{ABC}$, $\sigma = \rho_{AB} \otimes \frac{I_C}{d_C}$, $O = I_{AB} \otimes P_C$, where P_C is a projector supported on C and d_C is the dimension of C . Note

$$\frac{I_A}{d_A} \otimes \rho_{BC} = \frac{1}{d_A^2} \sum_{i=1}^{d_A^2} U_{A,i} \rho_{ABC} U_{A,i}^\dagger \quad (10)$$

for a set of unitaries $\{U_{A,i}\}$ that forms an orthogonal basis for $\mathcal{B}(\mathcal{H}_A)$. An example can be found in Ref. 13. Using joint convexity,

$$\text{Tr}\left(\frac{I_A}{d_A} \otimes \rho_{BC}(\hat{H}_B - \hat{H}_{BC})P_C\right) \leq \text{Tr}(\rho_{ABC}(\hat{H}_{AB} - \hat{H}_{ABC})P_C). \quad (11)$$

Left-hand side of the inequality is equal to $\text{Tr}(\rho_{ABC}(\hat{H}_B - \hat{H}_{BC})P_C)$. Since the inequality holds for all P_C ,

$$\text{Tr}_{AB}(\rho_{ABC}(\hat{H}_{AB} + \hat{H}_{BC} - \hat{H}_B - \hat{H}_{ABC})) \geq 0. \quad (12)$$

□

One may wish to find a similar inequality when partial trace is restricted to A or B . In both cases, the resulting operators are not even Hermitian.

III. CONCLUSION

We have proved an operator extension of strong subadditivity conjectured in Ref. 7. While our method was based on the operator convexity of the function $f(x) = x \log x$, Effros' approach can be generalized to other operator convex functions as well. It would be interesting to find applications of these new family of operator inequalities.

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